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LETTER TO THE EDITOR

# On equations with universal invariance

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**Abstract.** A general discussion of equations with universal invariance for a scalar field is provided in the framework of Lagrangian theory of first-order systems. The condition of universal invariance is that the Lagrange–Souriau form of the system is invariant up to a function factor with respect to these transformations. We consider in detail the case when this function is homogeneous in the first-order derivatives. When this function is non-trivial we have an example of non-Noetherian symmetries and obtain some equations intensively studied in the literature.

Recently there has been much interest in the study of partial differential equations which possess so-called universal invariance [1–4]. For a field with $N$ components, this means that if $\Phi^A$ $(A = 1, 2, \ldots, N)$ is a solution of the field equations, then $F \circ \Phi$ is also a solution of the same equation for any diffeomorphism $F \in \mathrm{Diff}(\mathbb{R}^N)$. One usually supposes that such an equation follows from a variational principle, i.e. is of the Lagrangian type.

The principle of universal invariance seems to produce many interesting equations of physical relevance. So it will be desirable to have a programme of classifying such equations following from the characterization above. We will start in this letter with the simplest case, namely when the Lagrangian is of first order and the field is scalar, i.e. $N = 1$.

In the case of first-order Lagrangians, one can use the formalism described in [5] which is very well suited to the study of Lagrangian systems with groups of symmetries. Applying this formalism, we will be able to write down a general equation with universal invariance for a scalar field.

In this letter, we present the general formalism for the case of a scalar field—in this case a complete discussion is possible—and derive the result announced above.

The geometric setting of the Lagrangian theory in particle mechanics is usually based on the Poincaré–Cartan 1-form, but it is also possible to use a 2-form with the Euler–Lagrange equations as the associated system [6]. This point of view was intensively exploited by Souriau [7] in connection with the Hamiltonian formalism. The proper generalization of these ideas to classical field theory is due to Krupka [8], Betounes [9, 10] and Rund [11]. We will follow the presentation from [5], but we will study directly, for simplicity, the case of a scalar field.

Let $S$ be a differentiable manifold of dimension $n + 1$. The first-order Lagrangian formalism is based on an auxiliary object, namely the bundle of 1-jets of $n$-dimensional submanifolds of $S$:

$$J_n^1(S) \equiv \cup_{p \in S} J_n^1(S)_p$$

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where $J_n^1(S)_p$ is the manifold of $n$-dimensional linear subspaces of the tangent space $T_p(S)$ at $S$ in the point $p \in S$. This manifold is naturally fibred over $S$. Let us denote by $\pi$ the canonical projection and construct a system of charts adapted to this fibred structure. We choose a system of local coordinates $(x^\mu, \psi)$ on the open set $U \subseteq S$; here $\mu = 1, \ldots, n$. Then on the open set $V \subseteq \pi^{-1}(U)$ we shall choose the local coordinate system $(x^\mu, \psi, \psi_\mu)$ defined as follows: if $(x^\mu, \psi)$ are the coordinates of $p \in U$ then the $n$-dimensional hyperplane in $T_p(S)$ corresponding to $(x^\mu, \psi, \psi_\mu)$ is spanned by the tangent vectors

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \psi_\mu \frac{\partial}{\partial \psi}. \tag{1}$$

We will systematically use the summation convention over the dummy indices.

By an evolution space we mean any (open) subbundle $E$ of $J_n^1(S)$. Note that $\dim(J_n^1(S)) = 2n + 1$.

Let us define, for any evolution space $E$,

$$\Lambda_{LS} \equiv \{\sigma \in \wedge^{n+1}(J_n^1(S)) | i_{Z_1} i_{Z_2} \sigma = 0, \forall Z_1, Z_2 \in \mathrm{Vect}(E) \text{ vertical}\}. \tag{2}$$

A vector field $Z \in \mathrm{Vect}(E)$ is vertical if, and only if, $\pi_* Z = 0$. It is clear that any $\sigma \in \Lambda_{LS}$ can be written in the local coordinates from above as follows:

$$\begin{aligned}\sigma = \varepsilon_{\mu_1,\ldots,\mu_n}(\sigma^{\mu_0}\,\mathrm{d}\chi_{\mu_0} \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} + n\sigma^{\mu_0\mu_1}\,\mathrm{d}\chi_{\mu_0} \wedge \delta\psi \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}) \\ + n!\tau\delta\psi \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.\end{aligned} \tag{3}$$

Here $\sigma^\mu$, $\sigma^{\mu_0\mu_1}$ and $\tau$ are smooth functions on $E$

$$\delta\psi \equiv \mathrm{d}\psi - \psi_\mu\,\mathrm{d}x^\mu \tag{4}$$

and $\varepsilon_{\mu_1,\ldots,\mu_n}$ is the signature of the permutation $(1, \ldots, n) \mapsto (\mu_1, \ldots, \mu_n)$.

One can verify this directly by performing a change of charts $(x^\mu, \psi) \mapsto (y^\mu, \zeta)$, inducing $(x^\mu, \psi, \psi_\mu) \mapsto (y^\mu, \zeta, \zeta_\mu)$, that the following equations have an intrinsic global meaning:

$$\sigma^\mu = 0 \tag{5}$$

$$\sigma^{\mu\nu} = \sigma^{\nu\mu}. \tag{6}$$

Any closed element $\sigma \in \Lambda_{LS}$ verifying (5) and (6) will be called a Lagrange–Souriau form on $E$ (LS-form). Such a $\sigma$ is of the form

$$\sigma = n\varepsilon_{\mu_1,\ldots,\mu_n}\sigma^{\mu_0\mu_1}\,\mathrm{d}\psi_{\mu_0} \wedge \delta\psi \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} + n!\tau\delta\psi \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n. \tag{7}$$

The closedness condition

$$\mathrm{d}\sigma = 0 \tag{8}$$

gives explicitly

$$\frac{\partial\sigma^{\mu\nu}}{\partial\psi_\rho} = \frac{\partial\sigma^{\mu\rho}}{\partial\psi_\nu} \tag{9}$$

$$\frac{\delta\sigma^{\mu\nu}}{\delta x^\nu} + \frac{\partial\tau}{\partial\psi_\mu} = 0. \tag{10}$$

We will call (6), (9) and (10) the structure equations.

A Lagrangian system over $S$ is a couple $(E, \sigma)$, where $E \subseteq J^1_n(S)$ is an evolution space over $S$ and $\sigma$ is a Lagrange–Souriau form on $E$. There is a natural equivalence relation between two such systems, $(E_1, \sigma_1)$ and $(E_2, \sigma_2)$, over the same manifold $S$, i.e. one must have a map $\alpha \in \mathrm{Diff}(S)$ such that $\dot{\alpha}(E_1) = E_2$ and

$$(\dot{\alpha})^* \sigma_2 = \sigma_1 \tag{11}$$

where $\dot{\alpha} \in \mathrm{Diff}(J^1_n(S))$ is the natural lift of $\alpha$. An evolution is an immersion $\Psi : M \to S$, where $M$ is some $n$-dimensional manifold (the 'spacetime' manifold).

Let $(E, \sigma)$ be a Lagrangian system over $S$; one says that $\Psi : M \to S$ verifies the Euler–Lagrange equations if

$$(\dot{\Psi})^* i_Z \sigma = 0 \tag{12}$$

for any vector field $Z \in \mathrm{Vect}(E)$. Here $\dot{\Psi} : M \to J^1_n(S)$ is the natural lift of $\Psi$. In local coordinates, one can arrange $\Psi$ such that it has the form $x^\mu \mapsto (x^\mu, \Psi(x))$; then $\dot{\Psi} : M \to J^1_n(S)$ is given by $x^\mu \mapsto (x^\mu, \Psi(x), (\partial\Psi/\partial x^\mu)(x))$ and (12) has the local expression

$$\sigma^{\mu\nu} \circ \dot{\Psi} \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu} - \tau \circ \dot{\Psi} = 0. \tag{13}$$

A interesting consequence of this equation is the following lemma.

*Lemma.* The Euler–Lagrange equations are trivial iff $\sigma = 0$.

We now come to the notion of symmetry. By a symmetry of the Euler–Lagrange equations, we understand a map $\phi \in \mathrm{Diff}(S)$ such that if $\Psi : M \to S$ is a solution of these equations, then so is $\phi \circ \Psi$.

In the case of a scalar field, one can completely describe the structure of a symmetry. We obtain the following theorem.

*Theorem 1.* Let $(E, \sigma)$ be a Lagrangian system for a scalar field and $\phi \in \mathrm{Diff}(S)$ a symmetry. There then exists $\rho \in \mathcal{F}(E)$ such that

$$(\dot{\phi})^* \sigma = \rho \sigma. \tag{14}$$

The function $\rho$ must satisfy the equation

$$\mathrm{d}\rho \wedge \sigma = 0 \tag{15}$$

or, in local coordinates,

$$\tau \frac{\partial \rho}{\partial \psi_\mu} + \sigma^{\mu\nu} \frac{\delta \rho}{\delta x^\nu} = 0 \tag{16}$$

$$\sigma^{\mu\nu} \frac{\partial \rho}{\partial \psi_\lambda} - \sigma^{\mu\lambda} \frac{\partial \rho}{\partial \psi_\nu} = 0. \tag{17}$$

*Proof.* Because $Z$ in (12) is arbitrary, one easily discovers that $\phi$ is a symmetry iff

$$(\dot{\Psi})^* i_Z\sigma = 0 \Longrightarrow (\dot{\Psi})^*(\dot{\phi})^* i_Z\sigma = 0 \qquad \forall Z \in \mathrm{Vect}(E) \qquad \forall \Psi : M \mapsto S.$$

We denote for simplicity

$$\tilde{\sigma} \equiv (\dot{\phi})^*\sigma. \tag{18}$$

One can show immediately that $\tilde{\sigma}$ is an LS-form and, therefore, it has the structure given by (7) with $\sigma^{\mu\nu} \mapsto \widetilde{\sigma^{\mu\nu}}$ and $\tau \mapsto \tilde{\tau}$. It follows from above that we have

$$\sigma^{\mu\nu} \circ \dot{\Psi} \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu} - \tau \circ \dot{\Psi} = 0 \Longrightarrow \widetilde{\sigma^{\mu\nu}} \circ \dot{\Psi} \frac{\partial^2 \Psi}{\partial x^\mu \partial x^\nu} - \tilde{\tau} \circ \dot{\Psi} = 0 \tag{19}$$

(see (13)). Equivalently,

$$\sigma^{\mu\nu}\psi_{\{\mu\nu\}} - \tau = 0 \Longrightarrow \widetilde{\sigma^{\{\mu\nu\}}}\psi_{\{\mu\nu\}} - \tilde{\tau} = 0 \tag{20}$$

where $\psi_{\{\mu\nu\}}$ is an arbitrary real symmetric matrix. In fact, it is more appropriate to consider expressions of the type appearing in (20) as functions of $J^2_n(S)$.

It is not hard to prove that (20) implies the existence of $\rho \in \mathcal{F}(E)$, such that

$$\widetilde{\sigma^{\mu\nu}}\psi_{\{\mu\nu\}} - \tilde{\tau} = \rho(\sigma^{\mu\nu}\psi_{\{\mu\nu\}} - \tau) \Longleftrightarrow \tilde{\tau} = \rho\tau \qquad \widetilde{\sigma^{\mu\nu}} = \rho\sigma^{\mu\nu}.$$

So, we find that

$$\tilde{\sigma} = \rho\sigma. \tag{21}$$

However, as previously noted, $\tilde{\sigma}$ is an LS-form, so $\rho\sigma$ must also be an LS-form. From the definition of an LS-form, it is clear that only the closedness condition (15) is missing. The derivation of (16) and (17) is elementary. □

*Remark 1.* If $\rho$ is not locally constant, then (15) implies that $\sigma$ is of the form

$$\sigma = \mathrm{d}\rho \wedge \omega \tag{22}$$

where $\omega$ is an $n$-form.

*Remark 2.* Let us suppose that the Lagrangian system $(E, \sigma)$ is non-degenerate, i.e.

$$\det(\sigma^{\mu\nu}) \neq 0. \tag{23}$$

This condition has an intrinsic global meaning as it easily follows from performing a change of charts. (The condition of non-degeneracy ensures that the Euler–Lagrange equations (13) can be 'solved' with respect to the second-order derivatives and the Cauchy problem can be well defined.) If we know (23) then one finds from (17) that $\partial\rho/\partial\psi_\lambda = 0$. Next, (16) gives $\partial\rho/\partial\psi = 0$ and $\partial\rho/\partial x^\mu = 0$, so $\rho$ is locally constant. This result is a sort of Lee-Hwa Chung theorem for the Lagrangian formalism (see, e.g., [12]).

*Remark 3.* The case $\rho = 1$ corresponds to the so-called Noetherian symmetries. For a detailed discussion see [5].

If a group $G$ acts on $S$: $G \ni g \mapsto \phi_g \in \mathrm{Diff}(S)$ then we say that $G$ is a group of (Noetherian) symmetries for $(E, \sigma)$ if, for any $g \in G$, $\phi_g$ is a (Noetherian) symmetry. In particular, we have

$$(\dot{\phi}_g)^* \sigma = \rho_g \sigma. \tag{24}$$

It is of physical interest to solve the following classification problem: given the manifold $S$ with an action of some group $G$ on $S$, find all the Lagrangian systems $(E, \sigma)$ where $E \subseteq J^1_n(S)$ is an open subset and $G$ is a group of (Noetherian) symmetries for $(E, \sigma)$. This goal will be achieved by simultaneously solving (6), (9), (10), (16), (17) and (24) in local coordinates and then investigating the possibility of globalizing the result.

We now explain the connection with the usual Lagrangian formalism. We can consider that the open set $V \subseteq \pi^{-1}(U)$ is simply connected by choosing it to be sufficiently small. From the structure equations (6), (9) and (10), one easily finds that there exists a (local) function $L$ on $V$ such that

$$\sigma^{\mu\nu} = \frac{\partial^2 L}{\partial \psi_\mu \partial \psi_\nu} \tag{25}$$

and

$$\tau = \frac{\partial L}{\partial \psi} - \frac{\delta}{\delta x^\mu}\left(\frac{\partial L}{\partial \psi_\mu}\right). \tag{26}$$

Now (13) takes the usual form for the Euler–Lagrange equations. $L$ is called a local Lagrangian. If $\sigma$ is given by (7), but with the coefficient functions as in (25) and (26) above, then we denote it by $\sigma_L$.

In our previously developed framework, $S = \mathbb{R}^n \times \mathbb{R}$ with global coordinates $(x^\mu, \psi)$, $(\mu = 1, \ldots, n)$. We can take $E = J^1_n(S) \simeq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ with global coordinates $(x^\mu, \psi, \psi_\mu)$. If $F \in \mathrm{Diff}(\mathbb{R})$, let us consider $\phi_F \in \mathrm{Diff}(S)$ given by

$$\phi_F(x, \psi) = (x, F(\psi)). \tag{27}$$

By definition, the Lagrangian system $(E, \sigma)$, defined previously, has universal invariance if $\phi_F$ is a symmetry (see (24)), i.e. we have

$$(\dot{\phi}_F)^* \sigma = \rho_F\, \sigma \tag{28}$$

for some functions $\rho_F \in \mathcal{F}(E)$, verifying (16) and (17). It is easy to show that $\rho_F$ must verify the following consistency relation:

$$\rho_{F_1} \circ \dot{\phi}_{F_2}\, \rho_{F_2} = \rho_{F1 \circ F_2} \tag{29}$$

which is easily recognized as a cocycle condition. One could be tempted to try to solve this cohomology problem. This can be achieved under some reasonable smoothness conditions, but we will prefer to circumvent this analysis.

We substitute (7) into (28) and obtain equivalently

$$(F')^2 \sigma^{\mu\nu} \circ \dot{\phi}_F = \rho_F \sigma^{\mu\nu} \tag{30}$$

$$F'(\tau \circ \dot{\phi}_F - F'' \sigma^{\mu\nu} \circ \dot{\phi}_F\ \psi_\mu \psi_\nu) = \rho_F\ \tau. \tag{31}$$

From (30), it easily follows that $\rho_F$ is a coboundary, i.e. is of the form

$$\rho_F = b \circ \dot{\phi}_F \, b^{-1} \tag{32}$$

for some function $b \in \mathcal{F}(E)$. In fact, one can show that the most general solution of (29) is of this type.

We will study, in fact, only a particular case which covers the equations from [1–4], namely when $b$ is a homogeneous function of degree $p \in \mathbb{N}$ in the variables $\psi_\mu$. Then, from (32), we have

$$\rho_F = (F')^p. \tag{33}$$

We insert (33) into (30) and (31) and consider that $F$ is an infinitesimal diffeomorphism, i.e.

$$F(\psi) = \psi + \theta(\psi) \tag{34}$$

with $\theta$ infinitesimal but otherwise arbitrary. We obtain

$$\frac{\partial \sigma^{\mu\nu}}{\partial \psi} = 0 \tag{35}$$

$$\psi_\lambda \frac{\partial \sigma^{\mu\nu}}{\partial \psi_\lambda} = (p-2)\sigma^{\mu\nu} \tag{36}$$

$$\frac{\partial \tau}{\partial \psi} = 0 \tag{37}$$

$$\psi_\lambda \frac{\partial \tau}{\partial \psi_\lambda} = (p-1)\tau \tag{38}$$

$$\sigma^{\mu\nu}\psi_\mu\psi_\nu = 0. \tag{39}$$

From the consistency equation (16), we obtain, for the case $p \neq 0$,

$$\sigma^{\mu\nu}\psi_\mu = 0 \tag{40}$$

so (39) is redundant in this case. Equation (17) is identically satisfied.

We analyse now, for the case $p \neq 0$, the system (35)–(38) and (40)†. First, we concentrate on the functions $\sigma^{\mu\nu}$. Let us note that (36) is the infinitesimal form of the homogeneity property

$$\sigma^{\mu\nu}(x, \lambda\psi_\mu) = \lambda^{p-2}\sigma^{\mu\nu}(x, \psi_\mu) \qquad \forall \lambda \in \mathbb{R}^*. \tag{41}$$

In the chart $\psi_0 \neq 0$, this means that $\sigma^{\mu\nu}$ is of the following form:

$$\sigma^{\mu\nu} = \psi_0^{p-2} s^{\mu\nu} \circ \pi \tag{42}$$

where $s^{\mu\nu}$ is a smooth function of the variables $x, y_1, \dots, y_{n-1}$ and we have defined

$$\pi(x, \psi_1, \dots, \psi_n) = \left(x, \frac{\psi_1}{\psi_0}, \dots, \frac{\psi_{n-1}}{\psi_0}\right). \tag{43}$$

† The case $p = 0$ can be analysed similarly.

From (6) we have

$$s^{\mu\nu} = s^{\nu\mu} \tag{44}$$

and from (40)

$$s^{00} = \sum_{i,j=1}^{n-1} y_i y_j s^{ij} \tag{45}$$

$$s^{0i} = -\sum_{j=1}^{n-1} y_j s^{ij}. \tag{46}$$

We still have structure equation (9) at our disposal. It is convenient to define the operator

$$D \equiv \sum_{j=1}^{n-1} y_j \frac{\partial}{\partial y_j}. \tag{47}$$

Then (9) is equivalent to

$$\frac{\partial s^{00}}{\partial y_i} = (p-2-D)s^{0i} \tag{48}$$

$$\frac{\partial s^{0i}}{\partial y_i} = (p-2-D)s^{ij} \tag{49}$$

$$\frac{\partial s^{ij}}{\partial y_k} = \frac{\partial s^{ik}}{\partial y_j}. \tag{50}$$

If we insert (46) into (49) we obtain

$$(p-1)s^{ij} = 0. \tag{51}$$

Analogously, if we insert (45) and (46) into (48) we obtain

$$(p-1)\sum_{j=1}^{n-1} y_j s^{ij} = 0. \tag{52}$$

If $p \neq 1$ then it easily follows that

$$\sigma = 0 \tag{53}$$

so we are left only with the case $p = 1$. In this case, (51) and (52) become identities and $s^{ij}$ are constrained only by (44) and (50). It follows that there exists a function $l$ depending on $x$ and $y_1, \ldots, y_{n-1}$ such that

$$s^{ij} = \frac{\partial^2 l}{\partial y_i \partial y_j}. \tag{54}$$

Then we obtain, from (45) and (46),

$$s^{i0} = -D\frac{\partial l}{\partial y_i} \tag{55}$$

and

$$s^{00} = (D^2 - D)l. \tag{56}$$

The structure of the functions $s^{\mu\nu}$ is completely elucidated. If we define

$$L_0 \equiv \psi_0 l \circ \pi \tag{57}$$

then it is elementary to prove that we have (25) with $L \to L_0$. If we define

$$\sigma' = \sigma - \sigma_{L_0} \tag{58}$$

then it is easy to analyse the structure of this auxilliary LS-form which also verifies invariance condition (28) and (33).

The final result can be summarized as follows.

*Theorem 2.* Let $(E, \sigma)$ be a first-order Lagrangian system for a scalar field having universal invariance (28) with $\rho_F$ given by (33). Then we have non-trivial solutions only for $p = 1$. In this case, we have

$$\sigma = \sigma_L. \tag{59}$$

For $p = 1$, we have

$$L = \psi_0 l \circ \pi + \psi\tau. \tag{60}$$

Here, $l$ is a smooth function depending on $x$ and $y_1, \ldots, y_{n-1}$ and $\tau$ is only $x$-dependent. The corresponding Euler–Lagrange equations (13), in the notation $\Psi_\mu \equiv \partial\Psi/\partial x^\mu$, $\Psi_{\{\mu\nu\}} \equiv \partial^2\Psi/(\partial x^\mu \partial x^\nu)$ and for the case when $l$ is $x$-independent, are

$$\sum_{i,j=1}^{n-1} \frac{\partial^2 l}{\partial y_i \partial y_j}\left(x, \frac{\Psi_i}{\Psi_0}\right) \Psi_0^{-3}(\Psi_i\Psi_j\Psi_{\{00\}} - \Psi_0\Psi_i\Psi_{\{0j\}} - \Psi_0\Psi_j\Psi_{\{0i\}} + \Psi_0^2\Psi_{\{ij\}}) - \tau = 0. \tag{61}$$

*Remark 4.* For $n = 2$, we obtain

$$\frac{\partial^2 l}{\partial y^2}\left(x, \frac{\Psi_1}{\Psi_0}\right) \Psi_0^{-3}(\Psi_1^2\Psi_{\{00\}} - 2\Psi_0\Psi_1\Psi_{\{01\}} + \Psi_0^2\Psi_{\{11\}}) - \tau = 0. \tag{62}$$

If we take $\tau = 0$, we obtain $(\cdots) = 0$ which is the equation appearing in [4].

*Remark 5.* Since $p = 1$, the universal invariance is not a Noetherian symmetry.

There are a number of results obtained in this letter which will be interesting to generalize.

First, one could try to extend this analysis to the case of a field with more than one component. This extension seems possible and plausible, but there might be some technical problems.

Next, we come to the universal invariance. Can one study the general case (32)? This seems to be a complicated problem.

Finally, one would like to generalize these results to higher-order Lagrangian systems. This problem is more manageable and some results in this direction will be reported soon. We will have to use a completely different method, because a generalization of our formalism to higher-order Lagrangian systems is not available.

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